How long does the quantum chaos last?

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The main purpose of this Comment is to point out that besides the short time scale of quantum chaos, confirmed once more in recent paper by Alicki et al, there is generally another time scale t_R, which is much longer and on which a partial quantum–classical correspondence persists. Namely, the quantum diffusion closely follows the classical one even though the former is dynamically stable. The absence of the long scale t_R in model studied by Alicki et al. is a result of a special choice for one model's parameters value.

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In a recent paper[1] Alicki, Makowiec and Miklaszewski presented one more confirmation for the existence of the short (logarithmic) time scale in quantum chaos using a simple model of kicked quantum top and a finite–time analogue of the classical dynamical Kolmogorov - Sinai (KS) entropy. This random time scale $t_r \sim \ln \hbar^{-1}$, on which the quantum motion is fully similar to the classical one including the exponential instability, had been discovered in Ref.[2] and was subsequently confirmed and further studied in many papers[3–5] (see also Ref.[6]).

The main purpose of this Comment is to point out that besides this short time scale there is generally another one, t_R, which is much longer ($\ln t_R \sim \ln \hbar^{-1}$) and on which a partial quantum–classical correspondence persists, namely, the quantum diffusion closely follows the classical one even though the former is dynamically stable [7] (see also Refs.[3,5]). The absence of the long scale t_R in model [1] is a result of a special choice for one model's parameters value.

Generally, the quantum top is described [8] by the unitary operator (per kick, $\hbar = 1$):

$$U(p,\, k,\, j) \,=\, \mathrm{e}^{-ikJ_z^2/2j}\, \mathrm{e}^{-ipJ_y} \tag{1}$$

which depends on two classical parameters, k and p, and one quantum parameter $j \gg 1$ (in quasiclassics). In Ref.[1] the value $p = \pi/2$ was chosen following Ref.[8] in which the only reason for such a choice was merely to simplify the quantum map. This particular choice leads to a nongeneric, fast ('ballistic') relaxation to the ergodic steady state. According to data in Fig.2 [1] the relaxation time $t_{er}(p) \approx 1.5$ iterations only, in this case. Moreover, some relaxation occurs even for almost regular motion ($k = 1$, see Fig.1 in Ref.[1]).

On the contrary, if $p \ll 1$ the relaxation becomes diffusive and relatively slow, and only for chaotic motion, of course, namely when parameter $K = pk > 1$. In the simplest case $|J_z| \ll j$ the diffusion rate in J_z is [8,9]

$$D \,\approx\, \frac{1}{2}\, (pj)^2\, C(K) \,\sim\, (pj)^2 \tag{2}$$

where $C(K) \sim 1$ accounts for dynamical correlations. Hence, the relaxation time (in number of kicks) is $t_{er}(p) \sim j^2/D \sim p^{-2} \gg 1$. During the relaxation process the quantum entropy keeps growing until it reaches the maximal value H_{er} for the ergodic state:

$$H(t) \,\equiv\, -\sum f(J_z,\, t)\, \ln f(J_z,\, t) \,\to\, H_{er} \,=\, \ln{(2j\,+\,1)} \tag{3}$$

where J_z are integers, and $f(J_z) = |\psi(J_z, t)|^2$ is the distribution function. An extra factor 2 in Ref.[1] (see Eq.(8) and Fig.1) for H_{er} is not completely clear but this is not the main point of our Comment, and will not be discussed here.

For an initially narrow Gaussian distribution the entropy in the diffusion regime is $H_D(t) \approx \ln{(2\pi \mathrm{e} Dt)}/2$ assuming $J_z \gg 1$ and $D(J_z) \approx \mathit{const}$ [9]. This entropy growth is much slower than that on the random time scale $t < t_r$ ($H_r(t) = t \cdot h_r$), and the corresponding KS entropy vanishes [10] as it should be for a quantum motion with discrete spectrum.

Notice that in the classical limit the entropy would grow indefinitely with constant rate $h_r = \Lambda \approx \ln{(K/2)}$ due to continuity of variable J_z as explained in the beginning of Ref.[1] (see also Ref.[11]). In the quantum case the classical instability h_r is restricted to the short time scale t_r which can be approximately found from the equation: $t \cdot h_r(t_r) = H_D(t_r)$. This gives a new asymptotic ($j \to \infty$) estimate $t_r \approx \ln{(pj)}/\ln{(pk)}$ in agreement with previous results [2–5].

The final steady state is ergodic with entropy (3) only under the additional condition [3,5,9] $t_{er} \ll t_H = (2j + 1)/2\pi = \exp{(H_{er})}/2\pi$ or $jp^2 \gg 1$ where t_H is the mean quasienergy level density also called the Heisenberg time. In the opposite case ($jp^2 \lesssim 1$) the quantum diffusion is restricted to the relaxation (diffusion) time scale[3,5,9]

$$t_R \,\sim\, D \,\sim\, (pj)^2 \lesssim t_H \tag{4}$$

Hence, the quantum steady state is essentially nonergodic due to localization of quantum diffusion. Assuming ap-

proximately exponential localization with a characteristic length $l \approx D$ the final steady state entropy in this case is $H_l \approx 1 + \ln D \to 2\ln(pj) \lesssim H_{er}$

The diffusive time scale t_R (4), which is the main point of our Comment, is always much longer as compared to the instability scale t_r. Only for $t \gg t_R$ the motion is completely dominated by the quantum effects.

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